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The diffusion–reaction (D–R) Hamiltonian and the solutions of certain types of linear and nonlinear D–R equations in one dimension

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Abstract
With a view to having further insight into the mathematical content of the non-Hermitian Hamiltonian associated with the diffusion–reaction (D–R) equation in one dimension, we investigate (a) the solitary wave solutions of certain types of its nonlinear versions, and (b) the problem of real eigenvalue spectrum associated with its linear version or with this class of non-Hermitian Hamiltonians. For case (a) we use the standard techniques to handle the quadratic and cubic nonlinearities in the D–R equation whereas for case (b) a newly proposed method, based on an extended complex phase space, is employed. For a particular class of solutions, an Ermakov system of equations is also found for the linear case. Further, corresponding to the 'classical' version of the above one-dimensional complex Hamiltonian, an equivalent integrable system of two, two-dimensional, real Hamiltonians is suggested.

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1. Introduction

In mathematical sciences some equations are highly privileged in the sense that their incarnations in analogous forms [1], at times of course with different meanings of the underlying symbols, explain altogether different phenomena in Nature. One such equation, besides the equation of continuity, is the diffusion–reaction (D–R) equation which has offered explanations of many phenomena lying in the domains of not only physics and chemistry but also biology and (now) perhaps social and economic sciences. While the linear version of the D–R equation under its various names such as the heat equation, Fokker–Planck equation, Schrödinger-like equation, etc has been studied very extensively in different contexts, the study of its nonlinear versions and sometimes in higher dimensions has also evoked considerable interest in recent years [2–8].

Before proceeding further some important remarks about the mathematical contents of the D–R and Schrödinger equations are worth making. The linear version of the D–R equation, namely

$$-D\nabla^2 C(\mathbf{x}, \mathbf{t}) + \mathbf{v}\cdot\nabla\mathbf{C}(\mathbf{x}, \mathbf{t}) + \mathbf{U}(\mathbf{x})\mathbf{C}(\mathbf{x}, \mathbf{t}) = -\frac{\partial\mathbf{C}(\mathbf{x}, \mathbf{t})}{\partial\mathbf{t}}, \tag{1}$$

can be compared with the time-dependent Schrödinger equation

$$\frac{-1}{2}\nabla^2\psi(\mathbf{x}, \mathbf{t}) + \mathbf{V}(\mathbf{x})\psi(\mathbf{x}, \mathbf{t}) = \mathbf{i}\frac{\partial\psi(\mathbf{x}, \mathbf{t})}{\partial\mathbf{t}}, \tag{2}$$

where $\hbar = m = 1$. In equation (1) D is the diffusion coefficient and the velocity $\mathbf{v}$ in general is a function of $\mathbf{x}$ and $\mathbf{t}$. In the present work, however, we shall consider $\mathbf{v}$ as a constant, independent of both space and time. Note that in mathematical literature equations (1) and (2) are classified differently in view of the different long-time behaviour of their solutions. As a matter of fact, in the limit $t \to \infty$, while the complete solution of (1) vanishes, the solution of (2) remains periodic in time. The presence of the velocity term in (1) further makes it distinct from (2). In many applications (listed in [3–5]) of (1) (particularly in some biological studies such as the spread of a favoured gene, population growth of species, ecological competition, etc), one investigates the dynamics and equilibrium properties of certain model systems as a combination of both diffusion and convection with some kind of back reaction (advection) [4]. The system thus involves velocity dependence and is a non-conservative one. In some cases, the description of the underlying phenomenon also requires an account of nonlinear terms (cf [3] and [6]). Other physical concepts attributed to the velocity term in (1) are those of turbulent diffusion or of anomalous diffusion in random media. The presence of velocity term in (1), in fact, makes the corresponding 'Hamiltonian' non-Hermitian and as a result the problem of reality of eigenvalues of such a Hamiltonian becomes of considerable interest in view of some recent studies [9–11] in this direction. We shall address some of these issues in the present work. Finally, while equation (2) in quantum mechanics is set on the basis of some physical requirements with a rich physics content in it, equation (1), on the other hand, is just a classical one like any other partial differential equation in mathematics, of course with different contextual meanings of various symbols.

In the present work, we shall investigate the solutions of the one-dimensional linear D–R equation,

$$C_t + vC_x = DC_{xx} - V(x)C, \tag{3}$$

and some of its nonlinear versions, namely

$$C_t + vC_x = DC_{xx} + [a + U(x)]C - b|C|C, \tag{4}$$

and

$$C_t + vC_x = DC_{xx} + [a + U(x)]C - b|C|^2C, \tag{5}$$

where a and b are real constants. We were motivated to study these equations from the recent works of Moiseyev and Gluck [5] and of Nelson and Shnerb [3]. These authors study the three-dimensional version of these equations with reference to the delocalization problem in population biology—a feature of the non-Hermitian character of the Hamiltonian associated with these equations. Undoubtedly, the present study of one-dimensional systems will have limited scope as far as the applications to physical problems are concerned; nevertheless it will provide some clue towards a better understanding of these 'real-world' problems in mathematical terms. While equation (4) can be considered as a particular type of generalization of the Malthus–Verhulst growth model [6] in the study of biological systems, equation (5) is the analogue of nonlinear Schrödinger equation. These cases, to the best of our knowledge, have not been studied earlier. In fact, equation (4) is a slightly different version of the equation studied by Nelson and Shnerb [3] in the sense that the quadratic nonlinearity now appears as $|C|C$ instead of C^2.

With regard to equation (3), we recast it in the form

$$HC(x,t) = -\frac{\partial C(x,t)}{\partial t}, \tag{6}$$

where H, a non-Hermitian 'Hamiltonian' operator, is given by [5]

$$H = -D\frac{\partial^2}{\partial x^2} + v\frac{\partial}{\partial x} + V(x).$$

To be more specific, one writes the solution of (3) as $C(x,t) = \sigma(x)\tau(t)$ and uses the method of separation of variables to obtain the eigenvalue equations,

$$\mathcal{H}\sigma(x) = \lambda\sigma(x), \tag{7}$$

with

$$\mathcal{H} = -D\frac{\mathrm{d}^2}{\mathrm{d}x^2} + v\frac{\mathrm{d}}{\mathrm{d}x} + V(x), \tag{8}$$

and $(\mathrm{d}\tau/\mathrm{d}t) = -\lambda\tau$. The solution of the latter equation implies an exponential decay of the complete solution with time. Note that for a complex eigenvalue λ there is a possibility of retaining the periodic behaviour of solutions with time—a feature built already in the Schrödinger equation (2). Further, we write the 'classical' analogue (using $p = -\mathrm{i}\,\mathrm{d}/\mathrm{d}x$) of $\mathcal{H}$ as

$$H(x,p) = Dp^2 + \mathrm{i}vp + V(x). \tag{9}$$

Note that for a symmetric potential the non-Hermitian Hamiltonian (9) is symmetric under parity ($x \to -x$) operation (P) and it is not symmetric under time-reversal ($t \to -t$, $\mathrm{i} \to -\mathrm{i}$) operation ($T$) or under the combined operations of parity and time reversal (PT). Therefore, in the light of the conjecture/prescription suggested by Bender *et al* [9] and used by others [10] equation (7) cannot admit real eigenvalues, as $\mathcal{H}$ is not a PT-symmetric Hamiltonian though it is non-Hermitian. For the case of a symmetric potential $V(x)$, we shall investigate this problem of reality of eigenvalues here within the framework of a complex phase-space approach proposed [12] and used recently [13] for a variety of complex potentials. In this approach, the real (x, p) phase plane is extended to a complex phase space characterized by writing x and p as [14]

$$x = x_1 + \mathrm{i}p_2, \qquad p = p_1 + \mathrm{i}x_2, \tag{10}$$

where (x_1, p_1) and (x_2, p_2) turn out to be the canonical pairs in an equivalent real two-dimensional space [15]. The arrangement of the paper is as follows: in section 2 we investigate the solitary wave solutions of the nonlinear equations (4) and (5). We study the eigenvalue problem associated with the non-Hermitian operator (8) in section 3 and highlight some other, so far unexplored, mathematical features of the Hamiltonian (8) (or (9)). In particular, for the linear case, in analogy with the Schrödinger equation [16] an Ermakov system [17] of equations is derived in this section. Also, corresponding to the one-dimensional complex Hamiltonian (9), an equivalent system of two, two-dimensional real Hamiltonians is obtained for an analytic potential function $V(x)$. Finally, concluding remarks are made in section 4.

2. Solitary wave solutions of the nonlinear D–R equations

In this section we obtain solitary wave solutions of the nonlinear equations (4) and (5).

2.1. Solution of equation (4)

For the travelling wave solution of equation (4) we consider the case when the random potential $U(x)$ is constant, say U_0, and for the solution $C(x,t)$ we make an ansatz

$$C(x,t) = r(\xi)\exp[\mathrm{i}\theta(\xi) + \delta t], \tag{11}$$

where $\xi = x - wt$, and δ and w are arbitrary real constants to be determined later. Using (11) in (4) and then separating the real and imaginary parts of the resultant expression, one obtains

$$(v - w)r' + r\delta = Dr'' - Dr\theta'^2 + (a + U_0)r - br^2, \tag{12}$$

$$(v - w)r\theta' = D(2r'\theta' + r\theta''). \tag{13}$$

Here the primes indicate the derivatives with respect to the variable ξ. Next we look for the solutions of these coupled total differential equations in $r(\xi)$ and $\theta(\xi)$ under some simplifying assumptions.

After defining $y = r^2\theta'$ for the right-hand side, equation (13) can easily be recast in the form, $y' = ((v - w)/D)y$, which admits a solution $y = y_0\exp[((v - w)/D)\xi]$, or

$$\theta' = (k/r^2)\exp[((v - w)/D)\xi], \tag{14}$$

where y_0 (or k) is the integration constant. For simplicity, we concentrate here on the case when $w = v$, i.e., when the parameter w in ansatz (11) becomes the convective velocity v of the system. This leads to $\theta' = k/r^2$.

For the above choices, equation (12) can be expressed as

$$r'' = \frac{k^2}{r^3} + \frac{1}{4}Br + \frac{3}{8}Ar^2,$$

which can easily be integrated to give

$$(r')^2 = -\frac{k^2}{r^2} + \frac{1}{4}Br^2 + \frac{1}{4}Ar^3 + \frac{k_1}{4}, \tag{15}$$

where $(k_1/4)$ is the constant of integration and $B = 4(\delta - a - U_0)/D$, $A = (8b/3D)$. Alternatively, by defining $S = r^2$ we write equation (15) in terms of the variable S as

$$(S')^2 = A\sqrt{S^5} + BS^2 + k_1 S - 4k^2. \tag{16}$$

For the solitary wave solutions, we set $k_1 = k = 0$, thereby reducing equation (16) to the form $S' = S(A\sqrt{S} + B)^{1/2}$ or equation (15) to

$$r' = (1/2)r\sqrt{Ar + B}, \tag{17}$$

which can be integrated to give [18] $r(\xi)$ as

$$r(\xi) = -(B/A)\sec^2\left(\frac{\sqrt{-B}}{4}\xi + \xi_0\right),$$

for $B < 0$, and

$$r(\xi) = (B/A)\mathrm{cosech}^2\left(\frac{\sqrt{B}}{4}\xi + \xi_0\right),$$

for $B > 0$. Correspondingly, the solution of $\theta' = 0$ equation is taken as $\theta =$ constant (say θ_0). Finally, the solutions of (4) in view of (11) becomes

$$C(x,t) = -(B/A)\exp(\mathrm{i}\theta_0 + \delta t)\sec^2\left(\frac{\sqrt{-B}}{4}\xi + \xi_0\right), \qquad (B < 0), \qquad (18)$$

and

$$C(x,t) = (B/A)\exp(\mathrm{i}\theta_0 + \delta t)\operatorname{cosech}^2\left(\frac{\sqrt{B}}{4}\xi + \xi_0\right), \qquad (B > 0), \qquad (19)$$

where ξ_0 ($=x_0 - vt_0$) is the constant of integration and the same can be fixed from the initial conditions.

2.2. Solution of equation (5)

Following the same procedure as for equation (4) in the above subsection with ansatz (11), the imaginary part of the resultant expression, in the present case, will yield the same equation as equation (13). The real part however now becomes

$$(v - w)r' + r\delta = Dr'' - Dr\theta'^2 + (a + U_0)r - br^3.$$

An equation analogous to equation (16) for the present case can be derived as

$$(S')^2 = AS^3 + BS^2 + k_1 S - 4k^2. \qquad (20)$$

Under the same simplifying assumptions as made in the above subsection for the case of solitary wave solutions, the solution of $\theta' = 0$ equation is again taken as $\theta = \theta_0$. The equation analogous to equation (17) now becomes

$$r' = (1/2)r\sqrt{Ar^2 + B}, \qquad (21)$$

where B is the same as before but $A = (2b/D)$. Equation (21) can be solved [18] to give

$$r(\xi) = \sqrt{-B/A}\sec\left(\frac{\sqrt{-B}}{2}\xi + \xi_0\right), \qquad (B < 0),$$

$$r(\xi) = \pm\sqrt{B/A}\operatorname{cosech}\left(\frac{\sqrt{B}}{2}\xi + \xi_0\right), \qquad (B > 0).$$

Finally, the solitary wave solution of equation (5) can be written as

$$C(x,t) = \sqrt{-B/A}\exp(\mathrm{i}\theta_0 + \delta t)\sec\left(\frac{\sqrt{-B}}{2}\xi + \xi_0\right), \qquad (B < 0), \qquad (22)$$

$$C(x,t) = \pm\sqrt{B/A}\exp(\mathrm{i}\theta_0 + \delta t)\operatorname{cosech}\left(\frac{\sqrt{B}}{2}\xi + \xi_0\right), \qquad (B > 0). \qquad (23)$$

where ξ_0 is the constant of integration to be determined from the initial conditions.

3. Eigenvalue problem associated with equation (7)

3.1. General results

Since the Hamiltonian operator (8) is non-Hermitian, the eigenvalue λ in (7) need not be real. Further for a symmetric potential, $\mathcal{H}$ is also not a PT-symmetric (a relaxed case of non-Hermiticity) one and hence may not admit [9] real eigenvalues. Naturally, the conjecture of

Bender *et al* [9] for the reality of eigenvalues of PT-symmetric potentials (developed mainly in the context of the Schrödinger equation) is bound to show some limitations in this case. While the concept of pseudo-Hermiticity [11] may be worth attempting for the present case, we have however been pursuing [12, 13] an altogether different method to handle the complex Hamiltonian systems. From this point of view, our approach is quite general and the concept of an extended complex phase space defined by (10) is used. In what follows we investigate the solution of an analogous D–R equation (7), in the sense that x and p in it are now complex.

Note that $V(x)$ in (7) in general could be complex just as x, p and σ are. Thus, we write $V(x) = V_r(x_1, p_2) + \mathrm{i}V_i(x_1, p_2)$, $\sigma(x) = \sigma_r(x_1, p_2) + \mathrm{i}\sigma_i(x_1, p_2)$, $\lambda = \lambda_r + \mathrm{i}\lambda_i$, and

$$\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\partial}{\partial x} - \mathrm{i}\frac{\partial}{\partial p_2}, \qquad \frac{\mathrm{d}^2}{\mathrm{d}x^2} = \frac{\partial^2}{\partial x_1^2} - 2\mathrm{i}\frac{\partial^2}{\partial x_1 \partial p_2} - \frac{\partial^2}{\partial p_2^2},$$

and use them in (7) to give

$$-D\left[\sigma_{r,x_1x_1} - 2\mathrm{i}\sigma_{r,x_1p_2} - \sigma_{r,p_2p_2} + \mathrm{i}\sigma_{i,x_1x_1} + 2\sigma_{i,x_1p_2} - \mathrm{i}\sigma_{i,p_2p_2}\right] + v\left[\sigma_{r,x_1} - \mathrm{i}\sigma_{r,p_2} + \mathrm{i}\sigma_{i,x_1} + \sigma_{i,p_2}\right] \\ + V_r\sigma_r - V_i\sigma_i + \mathrm{i}V_r\sigma_i + \mathrm{i}V_i\sigma_r = \lambda_r\sigma_r - \lambda_i\sigma_i + \mathrm{i}\lambda_i\sigma_r + \mathrm{i}\lambda_r\sigma_i. \tag{24}$$

Now we equate the real and imaginary parts of this expression separately to zero. This leads to the following pair of coupled partial differential equations in σ_r and σ_i:

$$-D\left[\sigma_{r,x_1x_1} - \sigma_{r,p_2p_2} + 2\sigma_{i,x_1p_2}\right] + v\left[\sigma_{r,x_1} + \sigma_{i,p_2}\right] + V_r\sigma_r - V_i\sigma_i = \lambda_r\sigma_r - \lambda_i\sigma_i,$$
$$-D\left[-2\sigma_{r,x_1p_2} + \sigma_{i,x_1x_1} - \sigma_{i,p_2p_2}\right] + v\left[-\sigma_{r,p_2} + \sigma_{i,x_1}\right] + V_r\sigma_i + V_i\sigma_r = \lambda_i\sigma_r + \lambda_r\sigma_i.$$

Further use of the Cauchy–Riemann conditions for the analyticity of $\sigma(x)$, namely,

$$\sigma_{r,x_1} = \sigma_{i,p_2}, \qquad \sigma_{r,p_2} = -\sigma_{i,x_1}, \tag{25}$$

leads to simpler forms of these equations, namely

$$-4D\sigma_{r,x_1x_1} + 2v\sigma_{r,x_1} + V_r\sigma_r - V_i\sigma_i = \lambda_r\sigma_r - \lambda_i\sigma_i, \tag{26}$$

$$-4D\sigma_{i,x_1x_1} + 2v\sigma_{i,x_1} + V_r\sigma_i + V_i\sigma_r = \lambda_i\sigma_r + \lambda_r\sigma_i. \tag{27}$$

As for the case of Schrödinger equation [12], we make an ansatz here for the solution, namely,

$$\sigma(x) \equiv \sigma_r + \mathrm{i}\sigma_i = \mathrm{e}^{g(x)}, \qquad g(x) = g_r(x_1, p_2) + \mathrm{i}g_i(x_1, p_2), \tag{28}$$

which gives $\sigma_r(x_1, p_2) = \mathrm{e}^{g_r}\cos g_i$, $\sigma_i(x_1, p_2) = \mathrm{e}^{g_r}\sin g_i$ or $g_r = (1/2)\ln\left(\sigma_i^2 + \sigma_r^2\right)$, $g_i = \tan^{-1}(\sigma_i/\sigma_r)$. Now, after using (28) into (26) and rationalizing the resultant expression with respect to the orthogonal functions $\cos(g_i)$ and $\sin(g_i)$, we obtain the following pair of coupled partial differential equations:

$$g_{r,x_1x_1} + \left(g_{r,x_1}\right)^2 - \left(g_{i,x_1}\right)^2 - (v/2D)g_{r,x_1} + (1/4D)(\lambda_r - V_r) = 0, \tag{29}$$

$$g_{i,x_1x_1} + 2g_{r,x_1}.g_{i,x_1} - (v/2D)g_{i,x_1} + (1/4D)(\lambda_i - V_i) = 0. \tag{30}$$

Interestingly, the same set of equations are also arrived at if one rationalizes equation (27) using (28). Thus, for a given complex potential, equations (29) and (30) in which the original ansatz for $\sigma(x)$ is now transcribed into that for $g(x)$ can be solved to obtain the real and imaginary parts of the eigenvalue λ. Further, we shall consider the two situations—one when the parameter(s) of the potential $V(x)$ are real or the other when they are complex. In the following we demonstrate the applications of these general results to the case of a simple harmonic oscillator corresponding to these two situations.

3.2. Example of a complex harmonic oscillator potential

We first consider the case of a complex harmonic oscillator potential,

$$V(x) = ax^2 \tag{31}$$

where a is real. Using (10), we write $V_r(x_1, p_2) = a\left(x_1^2 - p_2^2\right)$, $V_i(x_1, p_2) = 2ax_1p_2$ and make the following ansatz for g_r and g_i which is consistent with the Cauchy–Riemann conditions:

$$\begin{aligned} g_r(x_1, p_2) &= \alpha_1 x_1 + \alpha_2 p_2 + \alpha_{20}\left(x_1^2 - p_2^2\right) + \alpha_{11}x_1p_2, \\ g_i(x_1, p_2) &= -\alpha_2 x_1 + \alpha_1 p_2 - \tfrac{1}{2}\alpha_{11}\left(x_1^2 - p_2^2\right) + 2\alpha_{20}x_1p_2, \end{aligned} \tag{32}$$

where α_i, α_{ij} are real constants to be determined later. Substitutions of (32) into (29) and (30) and subsequently the rationalization of the resultant expressions with respect to the powers of x_1, p_2 and (x_1p_2) yield a set of relations among α_i and α_{ij}. These equations can be solved for the unknown constants and for λ_r and λ_i using the method described in our earlier works [12, 13]. Thus, for potential (31), we obtain

$$\begin{aligned} \lambda_r &= \mp 2\sqrt{aD} + (v^2/4D), \qquad \lambda_i = 0, \\ g_r(x_1, p_2) &= (v/4D)x_1 \pm \sqrt{a/16D}\left(x_1^2 - p_2^2\right), \\ g_i(x_1, p_2) &= (v/4D)p_2 \pm \sqrt{a/4D}x_1p_2 \end{aligned} \tag{33}$$

or equivalently,

$$\sigma(x) = \exp\left[(v/4D)x \pm \sqrt{a/16D}x^2\right]. \tag{34}$$

Next, we consider the case when the parameter a in potential (31) is complex, $a = a_r + \mathrm{i}a_i$. In that case we have

$$V_r(x_1, p_2) = a_r\left(x_1^2 - p_2^2\right) - 2a_ix_1p_2, \qquad V_i(x_1, p_2) = a_i\left(x_1^2 - p_2^2\right) - 2a_rx_1p_2. \tag{35}$$

As before, the use of ansatz (32) in (29) and (30) now yields a different set of equations for α_i and α_{ij} after the rationalization of the resultant expressions. In this case the eigenvalues turn out to be

$$\lambda_r = \mp\sqrt{2D}a_+ + (v^2/4D), \qquad \lambda_i = \pm\sqrt{2D}a_-, \tag{36}$$

and the eigenfunction $\sigma(x)$ in terms of $g_r(x_1, p_2)$ and $g_i(x_1, p_2)$ as

$$\begin{aligned} g_r(x_1, p_2) &= (v/4D)x_1 + (a_+/4\sqrt{2D})\left(x_1^2 - p_2^2\right) - (a_-/2\sqrt{2D})x_1p_2; \\ g_i(x_1, p_2) &= (v/4D)p_2 + (a_-/4\sqrt{2D})\left(x_1^2 - p_2^2\right) + (a_+/2\sqrt{2D})x_1p_2, \end{aligned}$$

or equivalently,

$$\sigma(x) = \exp[(v/4D)x + (1/4\sqrt{2D})(a_+ + \mathrm{i}a_-)x^2], \tag{37}$$

where $a_\pm = \sqrt{|a| \pm a_r}$.

Before we highlight two special cases of these general results it can be seen that the eigenvalues for the non-Hermitian operator (8) are real as long as the potential parameter a is real. Once a becomes complex, then the complexity of the eigenvalue may arise as a result of $a_i \neq 0$.

Note that for the case when $D = (1/2)$ and $v = 0$, above results for the complex harmonic oscillator trivially reduce to those derived by solving the Schrödinger equation in extended complex phase space (cf [13], section 3). Further, if we set $a_i = 0$, $a_r = a = |a|$ (or $a_- = 0$, $a_+ = \sqrt{2a}$), then it is not difficult to see that results (36) and (37) for the complex coupling reduce to that for the case of real coupling (cf equations (33) and (34)).

Mainly for the sake of a comparison we present here the results for the case of real harmonic oscillator, $V(x) = ax^2$, obtained by solving (7) in a real phase plane. Again using the ansatz $\sigma(x) = \exp[g(x)]$, with $g(x) = \alpha x^2 + \beta x$ for the solution, it is not difficult to arrive at the following results for λ and $\sigma(x)$:

$$\lambda = \mp\sqrt{aD} + (v^2/4D), \tag{38}$$

$$\sigma(x) = \exp\left[(v/2D)x \pm \sqrt{a/4D}x^2\right]. \tag{39}$$

It can be seen that results (38) and (39) differ slightly from (33) and (34) in terms of numerical factors. This is mainly because of the requirement of analyticity of $\sigma(x)$ used in the case of complex phase space.

3.3. Ermakov analogue of D–R equation (7)

About 125 years ago, Ermakov for the first time demonstrated [17] the linkage between the solutions of certain type of differential equations via an integral invariant. This latter construct, now termed as 'Ermakov' (or 'Lewis') invariant in the context of classical mechanics, has played [19] an important role in the study of time-dependent systems and in the quantum [16] and other contexts [21, 22] several new interpretations of this mathematical construct have been sought. Here, since the variable x characterizes the space dimension, we shall call this construct the 'space invariant'.

For the present purpose, we rewrite equation (7) as

$$\sigma'' - \gamma\sigma' + q^2(x)\sigma = 0, \tag{40}$$

where

$$\gamma = (v/D), \qquad q^2(x) = (\lambda - V(x))/D,$$

and look for its solutions in the form (ansatz)

$$\sigma(x) = \phi(x)\exp[\mathrm{i}f(x)]. \tag{41}$$

Now, after using (41) in (40) and equating the real and imaginary parts separately to zero in the resultant expression, we obtain

$$\phi'' - (f')^2\phi - \gamma\phi' + q^2\phi = 0, \tag{42}$$

$$f''\phi + 2f'\phi' - \gamma\phi f' = 0. \tag{43}$$

As before, equation (43) after defining $y = \phi^2 f'$, can easily be recast in the form $y' = \gamma y$, whose solution now becomes $y = \kappa\exp(\gamma x)$, or

$$f' = \kappa\,\mathrm{e}^{\gamma x}/\phi^2, \tag{44}$$

where κ is the constant of integration. While the integration of (44) in the form

$$f(x) = \kappa\int^x (\mathrm{e}^{\gamma x'}/\phi^2(x'))\,\mathrm{d}x', \tag{45}$$

suggests a phase-amplitude connection in the present case, its use in (42) leads to a nonlinear differential equation,

$$\phi'' - \gamma\phi' + q^2\phi = \kappa^2\,\mathrm{e}^{2\gamma x}/\phi^3. \tag{46}$$

In order to derive the space invariant, multiply equation (40) by ϕ and equation (46) by σ and subtract the latter to give

$$(\phi\sigma'' - \phi''\sigma) + \gamma(\sigma\phi' - \sigma'\phi) = -\kappa^2\,\mathrm{e}^{2\gamma x}\sigma/\phi^3. \tag{47}$$

This expression, after using $2(\phi\sigma' - \phi'\sigma)$ as the integrating factor, can immediately be integrated to give the space invariant, K, in the form

$$K = \kappa^2(\sigma/\phi)^2 + \mathrm{e}^{-2\gamma x}(\phi\sigma' - \phi'\sigma)^2. \tag{48}$$

Note that structure (48) satisfies $(\mathrm{d}K/\mathrm{d}x) = 0$, and hence is termed as 'space invariant'. Clearly, it is a manifestation of the phase-amplitude connection (45). Although, K as such appears to be independent of the form of $V(x)$ but the fact is that the role $V(x)$ enters through (46) or for that matter via (40). The system of equations (40), (46) and (48) constitute an Ermakov system.

Another interesting aspect of the 'classical' version (9) of the Hamiltonian (8), which is complex even in the real phase plane, can be explored by using the prescription of our earlier work [15]. Note that in the complex phase space (10) this version of $\mathcal{H}$, when expressed as $H(x, p) = H_1(x_1, x_2, p_1, p_2) + \mathrm{i}H_2(x_1, x_2, p_1, p_2)$, becomes a function of two complex variables x and p and its real and imaginary parts turn out to be

$$H_1(x_1, x_2, p_1, p_2) = D\left(p_1^2 - x_2^2\right) - vx_2 + V_r(x_1, p_2), \tag{49}$$

$$H_2(x_1, x_2, p_1, p_2) = 2Dp_1x_2 + vp_1 + V_i(x_1, p_2), \tag{50}$$

where $V(x) = V_r(x_1, p_2) + \mathrm{i}V_i(x_1, p_2)$ is used. Next, we compute the Poisson bracket $[H_1, H_2]_{\mathrm{PB}}$ from

$$[H_1, H_2]_{\mathrm{PB}} = \frac{\partial H_1}{\partial x_1}\frac{\partial H_2}{\partial p_1} - \frac{\partial H_1}{\partial p_1}\frac{\partial H_2}{\partial x_1} + \frac{\partial H_1}{\partial x_2}\frac{\partial H_2}{\partial p_2} - \frac{\partial H_1}{\partial p_2}\frac{\partial H_2}{\partial x_2}. \tag{51}$$

It can be noted that for H_1 and H_2 given by (49) and (50) the Poisson bracket (51) vanishes, if and only if the real and imaginary parts of $V(x)$ satisfy the Cauchy–Riemann conditions, namely, $V_{r,x_1} = V_{i,p_2}$, $V_{i,x_1} = -V_{r,p_2}$. In other words, this implies the analyticity of $V(x)$. Further, vanishing of the Poisson bracket (51) also suggests that H_1 and H_2 are in involution and independent in the sense [15] that vectors $\nu_j = J\nabla_y H_j(y)$ for $j = 1, 2$ and $y = x_1, x_2, p_1, p_2$ turn out to be linearly independent for (49) and (50). Here J is the symplectic unit matrix. It may be mentioned that H_1 and H_2 as given by (49) and (50) fulfil all the other requirements listed in [15] for a biharmonic function or for an auto-Backlund transformation and thereby enabling the integrability of an equivalent system in two real dimensions [15].

4. Concluding discussion

From the point of view of learning more about the D–R equation, two slightly disconnected aspects of this equation are explored in this work. In the first part, we have studied the solitary wave solutions of the D–R equation with some specific types of quadratic and cubic nonlinearities, which, to the best of our knowledge, have not been investigated earlier in this context. With regard to the application of these results, it concerns the modelling part of the study of a nonlinear phenomenon. In fact, there appear now many situations in the fields of population biology or in different branches of physics and chemistry, where the results obtained here can be useful in offering the alternatives while accounting for the nonlinearity in such studies.

We have restricted ourselves only to the study of solitary wave solutions of equations (4) and (5) by way of ignoring certain terms in equations (16) and (20). This is done mainly for simplicity; otherwise one can also retain all the terms in these equations and integrate the resultant expressions to obtain the travelling wave solutions of more general type. In fact, it is not difficult to arrive at the cnoidal wave solution in case of equation (5) for certain choices of the parameters and in analogy with the KdV equation [20].

In spite of the fact that the Hamiltonian (8) is non-Hermitian, it does not admit complex eigenvalues in general (cf section 3.2). The complex eigenvalues can, however, be expected if the parameter(s) of the potential also becomes complex in addition to the phase space. In some sense this will bring the D–R equation closer to the Schrödinger equation in spite of the presence of the velocity term in the former, particularly with reference to the large-t behaviour of their solutions.

Two previously unexplored (to the best of our knowledge) aspects of the D–R Hamiltonian (8) are highlighted in section 3.3. These studies only hint to the richness of the mathematical content present already in the Hamiltonian (8) or in its 'classical' version (9). In analogy with other studies [16, 19, 21, 22], a couple of possible interpretations of the constructed space invariant K (cf equation (48)) can be re-emphasized here in the present context. Firstly, since K is a space invariant and involves the solutions of both equations (40) and (46), it can act as a geometric constraint with regard to the validity of these solutions. Secondly, as the invariant K is the manifestation of a particular type of phase-amplitude connection (45) via the nonlinear equation (46), its existence itself suggests [19, 22] a nonlinear superposition principle in which the solution of (46) is expressible in terms of two linearly independent solutions of (40). Further applications of some of these results to stellar structure studies are in progress.

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